Onset of criticality and transport in a driven diffusive system

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We study transport properties in a slowly driven diffusive system where the transport is externally controlled by a parameter p. Three types of behavior are found: For $p<p'$ the system is not conducting at all, for intermediate p a finite fraction of the external excitations propagate through the system, and for $p>p_c$ the system becomes completely conducting. For all $p>p'$ the system exhibits self-organized critical behavior. In the middle of this regime, at p_c, the system undergoes a continuous phase transition described by critical exponents. [S1063-651X(97)50403-X]

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Cascade events where a single external perturbation may amplify in an avalanche like dynamics are seen in a variety of controllable systems, such as nuclear reactors [1], avalanche diodes [2], and lasers [3]. In statistical physics one considers cascade or avalanche events in a certain type of slowly driven diffusive system. The prototype of such dynamics is the sandpile model of Bak, Tang, and Wiesenfeld (BTW) [4]. This model exhibits large scale fluctuations with avalanches similar to the examples mentioned above, the difference being that the slowly driven systems self-organize to the critical transfer ratio, which is needed to see avalanches of all sizes; models exhibiting this phenomena are called self-organized critical (SOC). Following this idea, a number of similar models have been proposed. In particular, Zhang [5] and Manna [6] proposed sandpile models where the local redistribution contained a random element. This defined another universality class than the original BTW model [7]. Recently, similar ideas have been used in the explicit modeling (see [8,9]) of the observed rice-pile dynamics in an experiment performed by Frette *et al.* [10]. In fact, the rice-pile models in [8,9] belong to a universality class which encompasses a variety of different self-organized critical models, including both earthquake models, interface depinning models [11], and the Manna model [12]. Discrete models in one dimension which do not fall into this class exist, namely, the deterministic models in [13] and their stochastic variants proposed in [14].

In the present paper, we investigate the onset and robustness of criticality in the above mentioned ''rice-pile'' class of models. To this end, we introduce a control parameter p in a model that displays SOC, and investigate the robustness of the critical state as function of this control parameter. The control parameter will influence the so called transfer ratio, J, which is defined as the probability that an added particle will be transported through the system. When p is large the ratio will be critical, and when it is small the process will come to an end abruptly. The model will be critical in a whole range of values of the control parameter. Further, in contrast to other SOC models with continuous parameters [15], the critical properties of our model do not depend on the control parameter as long as this is above a certain threshold point, p'.

One may consider various formulations of stochastic SOC models in one dimension. One is the rice-pile version where the particles perform a directed walk through the system, and another is the ''slope version'' where one only considers the equivalent redistribution of slopes (local stresses). As numerically demonstrated in [12], the latter is similar to the Manna model [6]. Here we present the model in the ''rice-pile'' language, since this allows for a direct counting of transport properties in terms of the number of particles that are absorbed or transported through the system. For physical applications one may consider the added particles as excitations.

The model we study is defined as a simple extension of the one introduced in [9]. Consider a lattice $[1,L]$ on which particles are added consecutively, to the column $i=1$. The results to be reported in the following have been obtained for $L=200$. The state of the system is defined in terms of the height profile $h(i)$ of the pile, or equivalently in terms of the slopes $z_i=h(i)-h(i+1)$. The dynamics of the model is defined in two steps, of which (1) only takes place when (2) has come to an end: (1) An avalanche is initiated by adding one particle to the column $i=1$. If $z_1>1$ the column $i=1$ is considered active. (2) During each step in the avalanche all columns i which are active can transfer one particle to the adjacent position $i+1$. The probability for each such toppling is p, and in case of a toppling at position i, then one particle is transferred from column i to $i+1$. For all toppled particles, one tests whether any of the new local slopes z_{i-1}, z_i, z_{i+1} fulfills $z_j>1$. If this is the case for a column j, then one particle at column j is considered active in the next step. The procedure is then repeated with the new active sites, until finally there are no more active sites. Then a new particle must be added to the first column $(i=1)$.

Notice that the model contains one parameter, p, which describes the toppling probability for slopes $z_i>z_c\equiv 1$. Thus, this toppling probability does not depend on the value of the slope when it exceeds z_c. As a result, the model allows for very steep slopes, and consequently also for very large storage of externally imposed particles. It is this open storage capacity which allows the model to display both subcritical and critical behavior.

As usual, when considering systems that are slowly

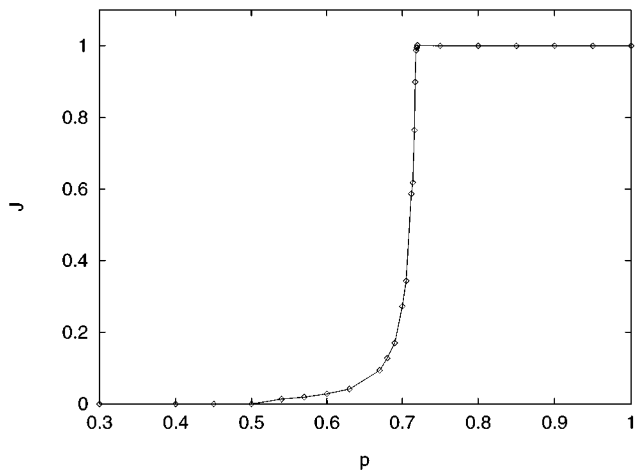

FIG. 1. Average transport ratio J of particles through the system, as function of the parameter p. Transport is defined in terms of fraction of conducted particles. Notice that when the average transport approaches unity, then in the rice-pile picture the growth velocity $(=1-J)$ of the pile approaches zero. The point where the transport starts is $p' \approx 0.538\,65$, and the point where it reaches unity is $p_c \approx 0.7185$.

driven, one has to run the dynamics for a certain transient time before its stationary features can be studied. In Fig. 1, we show the average transport ratio J, defined as the relative flux of outgoing particles to ingoing particles in the stationary state. This is equivalent to the probability that an added particle at position $i=1$ will escape at position $i>L$, as function of the parameter p. We observe three regimes: below $p' \approx 0.538\,65$ the system is nearly completely isolating and nothing penetrates. In the interval $[p', p_c]$, with $p_c \approx 0.7185$, the system is partially conducting with a slope $z = \langle z_i \rangle = [h(1) - h(L)]/L$, which fluctuates around a constant value determined by p since both $h(1)$ and $h(L)$ grow at the same rate in this regime. Finally, for $p > p_c$, the system is completely conducting, with a transport that exhibits the white noise behavior seen in Fig. 2.

Before entering the detailed discussion of the behavior of the system around the two transition points, p' where conduction starts and p_c where the motion of the pile gets pinned, we first investigate the fluctuations in the dynamics inside the pile. In Fig. 2 we show the fluctuations of $1-J$ as function of time for two p values, $p=0.6$ and $p=0.75$. For $p<p'$ (and large L), no particles ever reach $i=L$ (i.e., $J=0$). For higher p values, the transport out of the system is noisy, with a behavior that changes dramatically when passing $p=p_c$. In fact [see Figs. 2(a) and 2(b)], we would like to stress that although the dynamics in both intervals $[p', p_c]$ and $[p_c, 1]$ are conducting, the fluctuations in the transport properties have a phase transition at p_c: they exhibit a Brownian motion below p_c, but a white noise signal above it.

In Fig. 3 we plot the size of the relaxation events as measured by integrating over the number of topplings due to a single added particle. We see that for $p>p'$, the probability for having an avalanche of s topplings exhibit the power-law scaling

$$p(s) \propto 1/s^{\tau}, \quad \tau = 1.57 \pm 0.05. \tag{1}$$

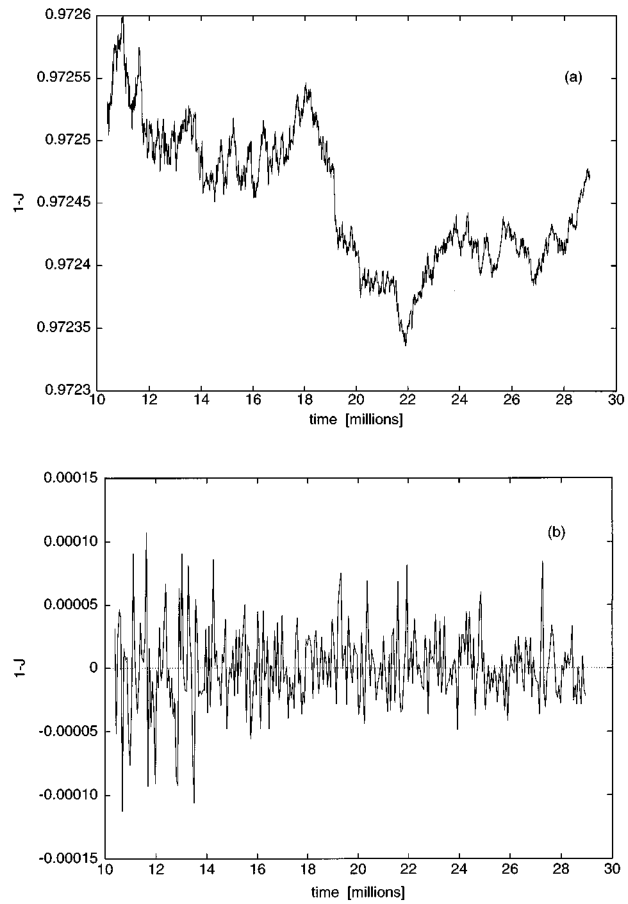

FIG. 2. Temporal dependence of the average absorbtion rate (=“velocity”) for (a) $p=0.6$ and (b) $p=0.75$. In both cases one observes fluctuations of about the same orders of magnitude. However, although the system is in the SOC state in both cases, the type of fluctuations are clearly different. For $p>p_c$ one observes Gaussian white noise, whereas $p<p_c$ leads to long-range correlations (reminiscent of Brownian motion) in the transmitted signal. For $p<p'$ everything gets absorbed, and there are no fluctuations.

This result is in accordance with what we expect for the rice-pile universality class for one-dimensional systems driven at the boundary [8,9,11,12]. Thus, the SOC state in our system is not affected by the possibility that some particles permanently are absorbed in the system, as is the case for $p'<p<p_c$. The type of nonconservation imposed in this interval of p values does not destroy scaling. For $p<p'$ the situation is different and the avalanche size distribution becomes exponentially bounded. When no particles are allowed to pass through the system, the system cannot build up long-range correlations.

In Fig. 4, we show the system properties at respectively the onset to conductivity $(p=p')$ and at the depinning transition $(p=p_c)$. The onset to conductivity is the simplest to understand. For low p the chance that excitations propagate through the system is small. The cascade triggered by adding one particle to the pile at $i=1$ dies out exponentially fast. Next, to understand the origin of the transition point $p=p'$ consider the cascade process at a point $i>1$. A particle toppling from position i makes particles at $i-1$, i and $i+1$ potentially active. If the corresponding slopes take values larger than z_c $(\equiv 1)$, each of the columns will topple with

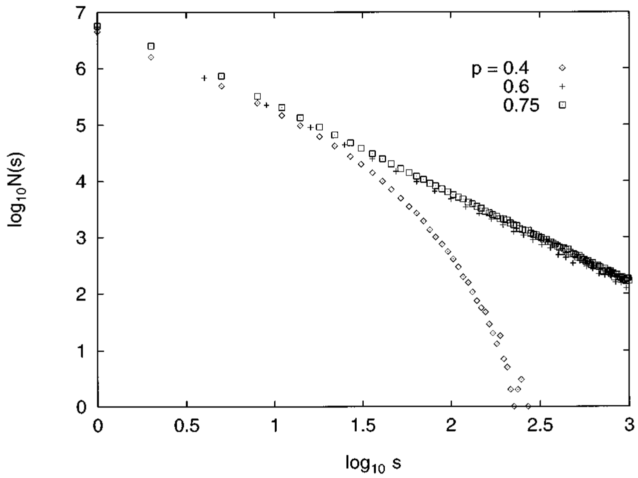

FIG. 3. (a) Log-log plot of the avalanche size distributions for $p=0.4$, $p=0.6$, and $p=0.75$. The observed power-law scalings correspond to the rice-pile universality class for $p>p'$, whereas the bounded avalanches for low p show that $p<p'$ does not give critical fluctuations.

probability p and make new columns active. In particular, when part of the pile is in a state with slopes $z_j \gg 1$ then the toppling events will always generate new active columns which topple exactly with probability p. This means that the spreading of active particles are determined by directed percolation (DP) on a square lattice with three descendents. This process has a critical point at $p=0.538\,65$ which is in complete agreement with our numerical estimate for p', thus we use $p'=0.538\,65$. Below the value p' the spreading of activity dies out and the z_i values increases indefinitely, making our subcritical DP picture self-consistent.

For $p>p'$, the DP becomes supercritical, and the system becomes at least partially conducting. Thus the local slopes do not grow indefinitely. Accordingly, the DP mapping breaks down, and the spreading of activity in the lattice will also be influenced by dynamically distributed absorbing states (states with slope 0 and 1) in the system. The transition at $p=p'$ may be quantified as in Fig. 4(a), where we show the vanishing of the current as function of the distance to p'. In order to obtain a better scaling we subtract the finite current J' at p', which occur only because we have a finite system. Given that $p'=0.538\,65$ is exactly known from DP studies, and measuring the finite current for a system of size $L=200$ at $p=p'$ to $J'\equiv J(0.538\,65)=0.013$, we observe numerically that $J(p)-J'\propto(p-p')^{\delta'}$ with $\delta'\approx 0.9$.

Approaching $p=p_c$, the system develops an ever increasing region extending from $i=1$ where many of the local slopes are small. At $p=p_c$, this region spans the entire system, and all particles added to $i=1$ will eventually be conducted out of the system at L (see Fig. 1). Above p_c the system profile remains bounded (pinned) at position $i=L$. By decreasing p, the profile starts moving upwards with a velocity that depends on p_c-p. In Fig. 4(b), we see that the number of particles absorbed in the system per unit time scales as

$$1-J\propto|p_c-p|^{\delta}, \quad \delta=0.9\pm 0.1. \tag{2}$$

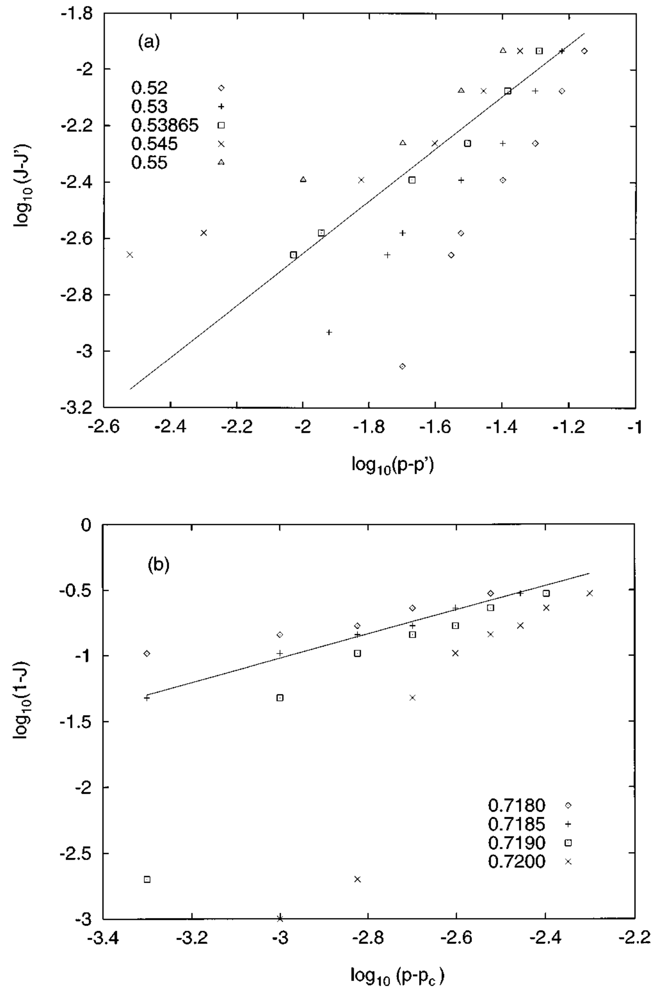

FIG. 4. Log-log plot of transport properties as a function of distance from the critical points: (a) The residual current $J(p)-J'$ vs $p-p'$ for p' values around $0.536\,85$ (the legend shows the p' values used). Due to the finite system, there is a small current $J'\equiv J$ ($0.536\,85$), and in order to improve the scaling behavior we subtract J' and find that the best scaling is indeed obtained for $p'=0.536\,85$. (b) The absorbtion $(=1-J)$ for various p_c values, cf. the legend. We find that the best scaling is obtained for $p_c=0.7185$.

In the language of rice-pile dynamics this would be the ''velocity'' of the pile $(=h(L)/t)$, and accordingly the transition at $p=p_c$ may be understood as a depinning transition for the profile of the rice pile. For p values on both side of this transition the system is critical, with exponents determined by the 'Manna' universality class (with avalanche exponent $\tau=1.55$ and avalanche dimension $D=2.25\pm 0.05$). It is an open question whether the depinning exponent δ can be related to the universal exponents of this very robust class of SOC models.

In summary, we have introduced and analyzed a slowly driven diffusive system which exhibits two phase transitions: the first described by the critical point of directed percolation, and it defines a transition to a self-organized critical state described by a very robust universality class. The second transition, at p_c, happens within the SOC state and can best be characterized as a transition to a state where the long-range meandering of the transport properties seen below p_c gets pinned at all values above p_c. We find it interesting that conductivity and the presence of a SOC state is intimately related in the present modeling.

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